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Renormalization-group analysis of the dilute Bose system in *d* dimensions at finite temperature

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Abstract

We study the d-dimensional Bose gas at finite temperature using the renormalization-group method. The flow-equations and the free energy are obtained for dimension d, and the cases $d < 2$ and $d = 2$ have been analysed in the limit of low and high temperatures. The critical temperature, the coherence length and the specific heat of a two-dimensional Bose gas are obtained using a regular solution for the coupling constant which is obtained from the scaling equations. The relevance of the dilute Bose gas for recent experimental results is also discussed.

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1. Introduction

The discovery of high-temperature superconductivity, the observation of the Bose–Einstein condensation (BEC) in ultracold trapped atomic gases and the discovery of the quasi-condensed state (QC) of polarized atomic hydrogen have created a wave of renewed interest in the physics of the dilute Bose gas, interacting via a weak repulsive potential. In the most general case of a parabolic dispersion law, the occurrence of the condensed phase at finite temperature depends on the dimensionality of the system, and it is well known that BEC in interacting uniform systems occurs only for $d > 2$. However, the absence of BEC does not necessarily imply the lack of a superfluid phase transition in $d = 2$, assuming that well-defined conditions are satisfied by the system [1].

At $T = 0$ the occurrence of a phase transition has been considered at $d = 2$ in a uniform Bose gas by Shick [2], Hines *et al* [3] using the t-matrix method [4], and for a d-dimensional system by Kolomeisky and Straley [5] using the renormalization-group (RG) method. At finite temperatures the $d = 2$ system has been studied using the t-matrix method by Chang and Fridberg [6], in the functional integral formalism by Popov [7], using RG method by Fisher and Hohenberg [8], and in a general diagrammatic formalism by Pieri *et al* [38, 39]. The case

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of a d-dimensional Bose gas at finite temperature was first mentioned by Weichman *et al* [9] related to critical fluctuations of superfluid helium.

The experimental results presented in [1] and the possible pair condensation in quasi-two-dimensional superconductors generated more complicated problems in connection with the condensation of the Bose gas in d dimensions at finite temperature. The purpose of this paper is to consider this problem using the RG method, starting from the normal phase. Following the method proposed in [5] we reconsider the results from [8] by solving more accurately the flow equations. New results are obtained for $d < 2$ and $d = 2$ and important physical quantities (the coherence length and specific heat) will be calculated as a function of temperature in the critical region.

In order to show the experimental support of such kind of calculations we briefly review the relevant experimental data and the existing theoretical models in section 2. Section 3 is devoted to the presentation of the model and the basic RG results at finite temperature for d-dimensional systems. In section 4 we solve the RG equations and calculate the density of the condensed number of particles and the coherence length as functions of temperature for $d < 2$. The important case of a $d = 2$ interacting Bose gas is treated in section 5. In the low temperature limit we used the regular Kolomeisky–Straley [5] solution to calculate the critical temperature, the free energy, the specific heat and the coherence length in the critical region. The high temperature limit is also discussed in this section. The importance of the present results and the development of the RG method for the realistic model of trapped systems are discussed in section 6. We also discuss in this section the connections of these results with the existing theoretical results from the literature and show the perspective of developing the RG method for $d \leqslant 2$ problems.

2. Relevant experimental data

In this section we present the basic experimental features for $d = 3$ and $d = 2$ Bose systems and analyse the stage of the theoretical models used in order to explain them.

2.1. Condensation in $d = 3$

The BEC has been observed in experiments involving vapour of alkali metals (rubidium, sodium and lithium) in which the atoms were confined in magnetic traps and cooled down to very low temperatures [10–16]. An important feature of these systems is that the trapped Bose gases are inhomogeneous and finite-size systems. In most of the cases the confining traps are approximated by harmonic potentials which provide a characteristic length scale for the system, defined by $a_h = [\hbar/(m\omega_h)]^{1/2}$ (ω_h is the trapping frequency). This specific length gives also the characteristic scale of the density variations. There is an important difference with respect to superfluid helium, where the variations of density are controlled by the microscopic length, fixed by the interatomic distance. The fact that the system is inhomogeneous has important consequences for the theoretical description because in this case BEC has to be studied in coordinate space, which is very different compared to the momentum space description. Another important fact is that these systems present important two-body interactions due to the atom–atom interaction. However, in order to compare the experimental data with the theoretical model we have to consider the BEC condensation taking also into consideration the harmonic trapping. This effect can change the physical parameters which can be measured. The inhomogeneity of these systems makes the many-body treatment non-trivial, but the dilute nature of the gas allows us to describe the system in terms of a single physical parameter, the s-wave scattering length, which is taken as the characteristic range of the interaction a.

The starting point for the theoretical models is a mean-field approach of inhomogeneous system, well known as the Gross–Pitaevskii theory [17–19]. The RG method was applied by Bijlsma and Stoof [20] and quantitative predictions were obtained about non-universal properties of this system. The critical exponent $\nu = 0.685$ calculated in [20] is in good agreement with $\nu = 0.67$ from the studies of the $O(2)$ model and with the recent result of Andersen and Strickland [21] obtained also by a more sophisticated RG method starting from the condensed state.

For the three-dimensional case, the effect of repulsive interaction on the condensation temperature (T_{BE}) at fixed density n has had a controversial history [20, 22, 23]. There have been two important discussions related to this subject. The first one was on the sign of the corrections introduced by the interaction term, both negative and positive values being obtained during the time. At the present stage, it is generally accepted that an increase in the condensation temperature will occur in the presence of repulsive interaction. The second dispute was connected to the exponents of the density and interaction range entering the condensation temperature ($T_c(a)$). The first prediction of $T_c(a)$ was $(T_c(a) - T_{BE})/T_{BE} \sim (na)^{2/3}$ [22], then $(T_c(a) - T_{BE})/T_{BE} \sim (na)$ [23] or by RG [20] $(T_c(a) - T_{BE})/T_{BE} \sim (na^3)^{1/6}$. Monte Carlo simulation [24, 25] predicted $(T_c(a) - T_{BE})/T_{BE} \sim an^{1/3}$, a result obtained also by a self-consistent approximation, the RG method for a N-vector model (in $N \to \infty$) by Baym *et al* [26, 27] and diagrammatic calculations [38, 39].

An important feature of all these experiments is related to the fact that the traps can be made very anisotropic. In this way quasi-two-dimensional systems can be studied and we will follow this discussion with these experiments.

2.2. Condensation in $d = 2$

It is well known that BEC condensation does not occur in two-dimensional systems. The absence of BEC in $d = 2$ interacting uniform Bose gas at finite temperature was proved by Popov [7] and Hohenberg [28]. However, for lower dimensional trapped Bose systems, such a transition may occur in the presence of harmonic trapping (which is in fact equivalent to modification of the density of shells) or in the presence of a power-law potential trapping [29]. The experiments with two-dimensional polarized hydrogen on a ^{4}He surface [30–32] showed that these data can be explained using the concept of quasi-condensate introduced by Popov [7] for a system of bosons which are locally coherent. This transition may appear due to interactions which suppress density fluctuations, and the two-dimensional bosonic system is described by a single wavefunction. The phase of this function fluctuates spatially, but the phase coherence persists on a finite length $L \sim \exp(n\lambda_T)$ where n is the $d = 2$ Bose gas density and $\lambda_T = \sqrt{2m\hbar^2/mk_B T}$ is the de Broglie thermal wavelength [28, 29, 33, 34]. For $T \to 0$ QC region $L \to \infty$ and the new phase becomes BEC at $T = 0$, which shows that in two dimensions it becomes a quantum phase transition. This idea is in agreement with the Kolomeisky–Straley [5] conjecture, that in a system of any finite size the transition exists and is determined by the scale of the system.

These experimental results [30–32] show a drastic change in the recombination rate of polarized hydrogen on the helium surface in the two-body collision. There is a characteristic temperature associated with a critical-like temperature, for which the second derivative of this rate has a discontinuity.

However, until now there have been no experimental data to be compared with the theoretical predictions made for results obtained for the critical behaviour.

3. Model and scaling equation

We consider the dilute Bose system in d dimensions at finite temperature T described by the action

$$S_{\text{eff}} = S_{\text{eff}}^{(2)} + S_{\text{eff}}^{(4)} \tag{1}$$

where

$$S_{\text{eff}}^{(2)} = \frac{1}{2}\sum_k \left[\frac{\hbar^2 k^2}{2m} - \mu - \frac{|\omega_n|}{\Gamma}\right] |\phi(k)|^2 \tag{2}$$

and

$$S_{\text{eff}}^{(4)} = \frac{u}{4}\sum_{k_1}\cdots\sum_{k_4}\phi(k_1)\cdots\phi(k_4)\delta\left(\sum_{i=1}^{4} k_i\right) \tag{3}$$

and the following notation has been used,

$$\sum_k \cdots \to k_{\text{B}}T\sum_n \int \frac{\mathrm{d}^d k}{(2\pi)^d}\cdots \tag{4}$$

$\omega_n = 2\pi\ nk_{\text{B}}\,T$ being the bosonic frequencies.

In equation (2) μ is the chemical potential of the bosonic system, described by the scalar field $\phi(k)$ and Γ is an energy parameter which controls the strength of the quantum fluctuations, and is initially $\Gamma = 1$, the classical limit being $\Gamma = 0$. Equation (3) represents the interaction between fluctuations, u ($u > 0$) being the coupling constant.

The propagator of the bosonic system has the form

$$G(k,\omega_n) = \left[\frac{\hbar^2 k^2}{2m} - \mu - \frac{|\omega_n|}{\Gamma}\right]^{-1} \tag{5}$$

and the renormalization transformations are carried out by integrating over a momentum shell and summing over all frequencies.

Regarding this procedure we mention that the summation over the bosonic frequencies (transformed to an integral) has to be done for all frequencies and not on a shell as in the method applied in [35, 36] for bosonic excitations. The physical argument is that in this case the presence of the chemical potential is important because we expect different contributions on the energy scale, which is in fact identical with a change of physical behaviour in different temperature domains. On the other hand, the limiting cases $\mu = \pm\infty$ correspond to the condensed and normal states, respectively.

After integrating out the degrees of freedom in the momentum shell, we rescale the variables as

$$k = \frac{k'}{b} \qquad \omega_n = \frac{\omega_n{}'}{b^z} \qquad T = \frac{T'}{b^z} \tag{6}$$

and the field operator as

$$\phi'(k',\omega_n') = b^{-(d+z+2)/2}\phi\left(\frac{k'}{b},\frac{\omega_n'}{b^z}\right) \tag{7}$$

where z is the dynamic critical exponent and we will introduce for b the parametrization $b = \mathrm{e}^l$ ($b > 0, l > 0$). The scaling equations obtained in [5, 8] for $T \neq 0$ and in [13] for $T = 0$ will be written as

$$\frac{\mathrm{d}\Gamma(l)}{\mathrm{d}l} = -(2 - z)\Gamma(l) \tag{8}$$

$$\frac{dT(l)}{dl} = zT(l) \tag{9}$$

$$\frac{d\mu(l)}{dl} = 2\mu(l) - K_d F_\mu[\mu(l), T(l), \Gamma(l)]\, u(l) \tag{10}$$

$$\frac{du(l)}{dl} = [4-(d+z)]u(l) - \tfrac{1}{4}K_d \left\{8F_{p-p}[\mu(l), T(l), u(l), \Gamma(l)] + 2F_{p-a}[\mu(l), T(l), u(l), \Gamma(l)]\right\} u^2(l) \tag{11}$$

$$\frac{dF(l)}{dl} = (d+z)F(l) + K_d F_f[\mu(l), T(l), u(l), \Gamma(l)] \tag{12}$$

where $F(l)$ is the free energy and $K_d = \pi^{-d/2}/(2^{d-1}\,\Gamma(d/2))$. The constants F_μ, F_{p-p}, F_{p-a} and F_f from equations (10)–(12) are given by the expressions

$$F_\mu = F_\mu[\mu(l), T(l), \Gamma(l)] = \frac{\Lambda^d \Gamma(l)}{\exp\left[\frac{\Gamma(l)}{k_B T(l)}\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)\right] - 1} \tag{13}$$

$$F_{p-a} = F_{p-a}[\mu(l), T(l), \Gamma(l)] = \frac{\Lambda^d \Gamma^2(l)}{2\Gamma(l)\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)} \coth\left[\frac{\Gamma(l)}{2k_B T(l)}\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)\right] \tag{14}$$

$$F_{p-p} = F_{p-p}[\mu(l), T(l)] = \frac{1}{4k_B T(l)} \frac{\Lambda^d \Gamma^2(l)}{\sinh^2\left[\frac{\Gamma(l)}{2k_B T(l)}\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)\right]} \tag{15}$$

$$F_f = F_f[\mu(l), T(l), \Gamma(l)] = k_B T(l) \ln\left\{1 - \exp\left[-\frac{\Gamma(l)}{k_B T(l)}\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)\right]\right\} - k_B T(l) \ln\left[1 - \exp\left(-\frac{\Gamma(l)}{k_B T(l)}\right)\right]. \tag{16}$$

Following [12, 13] we will solve these equations for $d < 2$ ($\epsilon = 2 - d$) and for the case $d = 2$ which is of special interest.

4. Below two dimensions

In order to solve the scaling equations (8)–(12) given in the previous section we start with equation (11) for the coupling constant at $T = 0$. Introducing the notation

$$C = \frac{mK_d \Lambda^{d-2}\Gamma^2(l)}{2\hbar^2} \tag{17}$$

this equation becomes

$$\frac{du(l)}{dl} = \epsilon u(l) - Cu^2(l) \tag{18}$$

with the initial condition $u(l = 0) = u_0$. The fixed point of equation (18) has the simple form

$$u^* = \frac{\epsilon}{C} \tag{19}$$

and the general solution

$$u(l) = \frac{u^*}{1 - \left(1 - \frac{u^*}{u_0}\right)\exp(-\epsilon l)}. \tag{20}$$

This solution satisfies the conditions

$$u(l=0)=u_0 \tag{21}$$

$$u(l\to\infty)=u^* \tag{22}$$

and in order to perform the calculation of physical quantities in the critical region we will use the linearized form of equation (20)

$$u(l)\simeq u^*+\frac{u^*}{u_0}(u_0-u^*)\mathrm{e}^{-\epsilon l} \tag{23}$$

In the limit of low temperatures, the chemical potential $\mu(l)$ given by equation (10) becomes

$$\mu(l)=-K_d\Lambda^d\mathrm{e}^{2l}\int_0^l \mathrm{d}l' \,\frac{u(l')\mathrm{e}^{-2l'}}{\exp\left[\frac{\hbar^2\Gamma(l')\Lambda^2}{2mk_\mathrm{B}T}\mathrm{e}^{-2l'}-\frac{\Gamma(l')\mu(l')}{2mk_\mathrm{B}T(l')}\right]-1}. \tag{24}$$

From equations (8) and (9) we obtain

$$\Gamma(l)=\Gamma\mathrm{e}^{-(2-z)l} \tag{25}$$

$$T(l)=T\mathrm{e}^{2l} \tag{26}$$

and using for $\mu(l)$ the lowest approximation ($\mu(l)=\mu\mathrm{e}^{2l}$) we calculate the second term from the exponential of equation (24) as

$$\frac{\Gamma\mu}{2mk_\mathrm{B}T}\ll 1. \tag{27}$$

This inequality is satisfied even in the low temperature domain, because Γ^{-1} is the energy scale of the quantum fluctuations and in this domain these are important. Using these considerations, equation (24) gives

$$\mu(l)\simeq -K_d\Lambda^d\mathrm{e}^{2l}\int_0^l \mathrm{d}l' \,\frac{\mathrm{e}^{-2l'}u(l')}{\exp\left(\frac{\hbar^2\Gamma\Lambda^2}{2mk_\mathrm{B}T}\mathrm{e}^{-2l'}\right)-1} \tag{28}$$

and in order to calculate the condensed density n and the coherence length ξ we will use equation (23) for $u(l)$. The renormalization procedure will be stopped for $l=l^*$, given by the equation

$$\mu(l^*)=-\alpha\frac{\hbar^2\Lambda^2}{2m} \tag{29}$$

where $\alpha\lesssim 1$. If we introduce the notations

$$a=\frac{\hbar^2\Lambda^2\Gamma}{2mk_\mathrm{B}T} \tag{30}$$

$$b=1+\frac{\epsilon}{2} \tag{31}$$

we obtain from equations (28) and (29)

$$\mathrm{e}^{-2l^*}\left(1+\frac{2\epsilon}{\alpha}\right)\simeq\frac{4\epsilon}{\alpha}I(l^*) \tag{32}$$

where

$$I(l^*)=\int_0^{l^*}\mathrm{d}l' \,\frac{\mathrm{e}^{-2bl'}}{\exp(a\mathrm{e}^{-2l'})-1}. \tag{33}$$

This integral can be expressed as

$$I(l^*)\simeq\frac{1}{2a^b}\ln\frac{1}{A}\left[1-\frac{\epsilon}{2}\frac{F(A)}{\ln\left(\frac{1}{A}\right)}\right] \tag{34}$$

where $x_0 = 2\epsilon$ and

$$F(A) = \int_{x_0}^{\infty} \mathrm{d}x \; x^{\frac{\epsilon}{2}-1} \ln(1 - \mathrm{e}^{-x}). \tag{35}$$

In the limit of low temperatures, equation (32) gives

$$\mathrm{e}^{-2l^*} = 2\epsilon \left[\frac{2mk_{\mathrm{B}}T\Gamma}{\hbar^2\Lambda^2}\right]^{1+\epsilon} \ln\left(\frac{1}{2\epsilon}\right). \tag{36}$$

Using these results we can calculate the density of the condensed bosons as

$$n = \mathrm{e}^{-dl^*} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\exp\left[\frac{1}{k_{\mathrm{B}}T(l^*)}\left(\frac{\hbar^2k^2}{2m} - \mu(l^*)\right)\right] - 1}. \tag{37}$$

The integral from equation (37), denoted by A,

$$A = K_d \int_0^{\infty} \mathrm{d}k \frac{k^{d-1}}{\exp\left[\frac{1}{k_{\mathrm{B}}T(l^*)}\left(\frac{\hbar^2k^2}{2m} - \mu(l^*)\right)\right] - 1} \tag{38}$$

will be transformed using the notation

$$x = \frac{\hbar^2k^2}{2mk_{\mathrm{B}}T(l^*)} - \frac{\mu(l^*)}{k_{\mathrm{B}}T(l^*)} \tag{39}$$

and we get

$$A = \frac{1}{2}K_d \left[\frac{2mk_{\mathrm{B}}T(l^*)}{\hbar^2}\right]^{(d/2)} \int_{|x_m|}^{\infty} \mathrm{d}x \; \frac{x^{(d-2)/2}}{\mathrm{e}^x - 1} \tag{40}$$

where $|x_m| = \mu(l^*)/k_B T(l^*)$. In order to perform the integral given by equation (40) we consider ϵ small (d close to $d = 2$) and in this case

$$A \simeq -\frac{1}{2}K_d \left[\frac{2mk_{\mathrm{B}}T(l^*)}{\hbar^2}\right]^{d/2} \ln\left[1 - \exp\left(-\frac{\mu(l^*)}{k_{\mathrm{B}}T(l^*)}\right)\right]. \tag{41}$$

Using $T(l^*) = T\mathrm{e}^{2l^*}$ and equation (29) we obtain the condensed density

$$n(T) \simeq \frac{1}{4\pi}\left[\frac{2mk_{\mathrm{B}}T}{\hbar^2}\right] \ln \frac{1}{2\epsilon \ln\left(\frac{1}{2\epsilon}\right)}. \tag{42}$$

The temperature dependence of the coherence length $\xi(T)$ will be calculated using

$$\xi^{-2}(T) = \frac{2m}{\hbar^2}|\mu(l^*)| \tag{43}$$

where $\mu(l^*)$ is given by equation (29). This gives

$$\xi(T) = \frac{1}{\Lambda\alpha^{1/2}}\mathrm{e}^{l^*} \tag{44}$$

and using equation (36)

$$\xi(T) = \frac{1}{\left(2\epsilon \ln\left(\frac{1}{2\epsilon}\right)\right)^{1/2}} \frac{\hbar}{(2mk_{\mathrm{B}})^{1/2}} T^{-\frac{1}{2}\left(1+\frac{\epsilon}{2}\right)} \tag{45}$$

if we take $\alpha \simeq 1$.

This behaviour appears as universal and is not surprising because, as was mentioned in [5], the first contribution to the calculation of chemical potential in the fixed point gives a first term as universal.

5. Two dimensions

The dilute Bose gas at $T = 0$ was studied in [5] and is known to be marginal. Here we consider the system at finite temperature in the low temperature approximation. The free energy and the specific heat will also be calculated for this system taking $z = 2$. The high temperature limit will also be analysed.

5.1. Low temperature limit

5.1.1. Scaling equations. In this case the scaling equations have the form

$$\frac{\mathrm{d}\Gamma(l)}{\mathrm{d}l} = 0 \tag{46}$$

$$\frac{\mathrm{d}T(l)}{\mathrm{d}l} = 2T(l) \tag{47}$$

$$\frac{\mathrm{d}u(l)}{\mathrm{d}l} = -\frac{mK_2}{2\hbar^2}u^2(l) \tag{48}$$

$$\frac{\mathrm{d}\mu(l)}{\mathrm{d}l} = 2\mu(l) - \frac{\Lambda^2 K_2 u(l)}{\exp\left(\frac{\hbar^2\Lambda^2}{2m k_\mathrm{B} T(l)}\right) - 1}. \tag{49}$$

Equations (46)–(48) have the solutions

$$\Gamma(l) = \Gamma \tag{50}$$

$$T(l) = T\mathrm{e}^{2l} \tag{51}$$

$$u(l) = \frac{4\pi\hbar^2}{m}\frac{1}{l + l_0} \tag{52}$$

where Γ will be taken as $\Gamma = 1$ and l_0 has been calculated as

$$l_0 = \frac{4\pi h^2}{mu_0}. \tag{53}$$

The solution of equation (49) has the form

$$\mu(l) = -\frac{4\Lambda^2\hbar^2}{2m}\mathrm{e}^{2l}\int_0^l \frac{\mathrm{d}l'}{l' + l_0}\frac{\mathrm{e}^{-2l'}}{\exp\left(\frac{\hbar^2\Lambda^2}{2m k_\mathrm{B} T}\mathrm{e}^{-2l'}\right) - 1} \tag{54}$$

which will be written as

$$\mu(l) = -\frac{4\Lambda^2 h^2}{2m}\mathrm{e}^{2l}\,\frac{2mk_\mathrm{B}T}{\mathrm{h}^2\Lambda^2}\left\{\frac{1}{2l_0}\ln\left[1 - \exp\left(-\frac{\hbar^2\Lambda^2}{2mk_\mathrm{B}T}\right)\right]\right.$$
$$\left. - \frac{1}{2l_0}\left(1 + \frac{l}{l_0}\right)^{-1}\ln\left[1 - \exp\left(-\frac{\hbar^2\Lambda^2}{2mk_\mathrm{B}T}\mathrm{e}^{-2l}\right)\right]\right\} - \frac{2mk_\mathrm{B}T}{h^2\Lambda^2}F(l) \tag{55}$$

where

$$F(l) = \int_0^{2l}\frac{\mathrm{d}x}{(x + 2l_0)^2}\ln\left[1 - \exp\left(-\frac{\hbar^2\Lambda^2}{2mk_\mathrm{B}T}\mathrm{e}^{-x}\right)\right]. \tag{56}$$

The renormalization procedure will be stopped at l^* defined also by

$$\mu(l^*) = -\alpha\frac{\hbar^2\Lambda^2}{2m}$$

but in this case $\mu(l)$ is given by equation (55). Following the same procedure we calculate

$$\mathrm{e}^{-2l^*} \simeq \frac{4}{\alpha}\left[\frac{4}{\alpha} - \ln\frac{4}{\alpha}\right]\frac{2mk_\mathrm{B}T}{\hbar^2\Lambda^2}\frac{1}{\ln\frac{\alpha}{4}\frac{\hbar^2\Lambda^2}{2mk_\mathrm{B}T}} \tag{57}$$

and if we introduce the effective temperature $T_0 = \hbar^2\Lambda^2\alpha/8mk_\mathrm{B}$ equation (57) will be written as

$$\mathrm{e}^{-2l^*} = \frac{C(\alpha)}{4}\frac{T}{T_0}\frac{1}{\ln\frac{T_0}{T}} \tag{58}$$

where

$$C(\alpha) = \frac{4}{\alpha}\left(\frac{4}{\alpha} - \ln\frac{4}{\alpha}\right).$$

Using equation (44) we get for the coherence length

$$\xi(T) \sim \left|\frac{\ln\frac{T_0}{T}}{\frac{T}{T_0}}\right|^{1/2}. \tag{59}$$

The number of condensed bosons near the critical temperature has been calculated using a similar relation with (37) and we get

$$n = \frac{2mk_\mathrm{B}T}{4\pi\hbar^2}\ln\frac{1}{1-\exp\left(-\frac{\hbar^2\Lambda^2}{2mk_\mathrm{B}T}\mathrm{e}^{-2l^*}\right)} \tag{60}$$

which gives in the low temperature limit

$$n = \frac{2mk_\mathrm{B}T}{4\pi\hbar^2}\ln\left(\ln\left(\frac{T_0}{T}\right)\right)$$

This equation can be inverted [8] and taking $\Lambda \sim 1/a$ (a is the range of the interaction u) we obtain the critical temperature of QC transition as

$$T_{2D} = \frac{\hbar^2}{2m}\frac{4\pi n}{\ln\left(\ln\left(\frac{1}{na^2}\right)\right)} \tag{61}$$

a result which agrees with the t-matrix calculations [3, 4], and also with the RG calculations from [5]. However, we mention that the method used in [8] is equivalent to the calculation of the singular part of the free energy in l^* and the singular behaviour of coupling constant makes the calculations very difficult.

5.1.2. The free energy and specific heat. The free energy given by the general equation (12) has the form

$$\frac{\mathrm{d}F(l)}{\mathrm{d}l} = 4F(l) + C_f(\mu(l), T(l)) \tag{62}$$

with

$$C_f = K_2\Lambda^2 k_\mathrm{B}T(l)\ln\left\{1-\exp\left[-\frac{\Gamma}{k_\mathrm{B}T(l)}\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)\right]\right\}. \tag{63}$$

Using the substitution $T(x) = T\mathrm{e}^{2x}$ we write the solution of equation (62) as

$$F(l) = \int_0^l \mathrm{d}x\,\mathrm{e}^{-4x}C_f(T\mathrm{e}^{2x}) \tag{64}$$

where we approximate C_f as

$$C_f = K_2\,\Lambda^2\,k_B T\,\ln\left[1-\exp\left(-\frac{\Gamma}{k_B T}-\mu(l)\right)\right]. \tag{65}$$

The expression for $F(l^*)$ becomes

$$F(l^*) = \frac{\Lambda^2}{2\pi}k_B T\int_0^{l^*}\mathrm{d}x\,\mathrm{e}^{-2x}\ln\left(1-\mathrm{e}^{-\frac{D}{T}\mathrm{e}^{-2x}}\right) \tag{66}$$

where

$$D = \frac{\Gamma}{k_B}\left(\frac{\hbar^2\Lambda^2}{2m}-\mu\right)$$

In the low temperature limit equation (4.21) will be approximated as

$$F = \frac{\Lambda^2}{4\pi}\,k_B T\,\ln\frac{D}{T}\left[1-\frac{\frac{T}{T_0}\frac{C(\alpha)}{4}}{\ln\frac{T_0}{T}}\right]+\frac{\Lambda^2}{4\pi}(k_B T)^2\,\frac{\frac{T}{T_0}}{\ln\left|\frac{T}{T_0}\right|}\,\ln\left[\frac{\ln\left|\frac{T}{T_0}\right|}{\left|\frac{T}{T_0}\right|}\right]$$

$$+\frac{\Lambda^2}{4\pi}k_B T\left[\frac{\frac{T}{T_0}}{\ln\left|\frac{T}{T_0}\right|}-1\right]. \tag{67}$$

From this equation we calculate the specific heat $C_v(T) = -T\partial^2 T/\partial T^2$ and the dominant contribution in temperature has the form

$$C_v(T) = C_0\frac{\left|\frac{T}{T_0}\right|}{\ln^3\left|\frac{T}{T_0}\right|} \tag{68}$$

where $C_0 = C_0(\Lambda)$ which shows that the result is Λ-dependent.

5.2. *High temperature limit*

The high temperature domain is called the 'classical' domain, because it is dominated by the classical fluctuations of the $\phi(x,\tau)$ and is usually defined by $z = 0$.

5.2.1. Scaling equations. From the general equations (8)–(11) we write these equations in the high temperature limit as

$$\frac{\mathrm{d}[T(l)\Gamma^{-1}(l)]}{\mathrm{d}l} = 2[T(l)\Gamma^{-1}(l)] \tag{69}$$

$$\frac{\mathrm{d}\mu(l)}{\mathrm{d}l} = 2\mu(l)-\frac{1}{2\pi}\tilde{F}_\mu[\mu(l),T(l)]v(l) \tag{70}$$

$$\frac{\mathrm{d}v(l)}{\mathrm{d}l} = 2v(l)-\frac{10}{4\pi}\tilde{F}_v[\mu(l),\ T(l)]v^2(l) \tag{71}$$

where

$$v(l) = k_B T u(l) \tag{72}$$

and $\tilde{F}_\mu,\ \tilde{F}_v$ have been obtained from equations (13)–(15) in the limit $T\to\infty$ as

$$\tilde{F}_\mu[\mu(l),\ T(l)] \simeq \frac{\Lambda^2}{\left[\frac{\hbar^2\Lambda^2}{2m}-\mu(l)\right]} \tag{73}$$

$$\tilde{F}_v[\mu(l),\ T(l)] \simeq \frac{\Lambda^2}{\left[\frac{\hbar^2\Lambda^2}{2m}-\mu(l)\right]^2}. \tag{74}$$

In order to solve equations (69)–(71) we have to define $\tilde{l}$, the value of l at which the flow enters the classical regime defined by

$$\frac{T(l)}{\Gamma(l)} \gg 1 \tag{75}$$

and introduce

$$\mu(\tilde{l}) = \tilde{\mu}_0 \qquad u(\tilde{l}) = \tilde{u}_0 \qquad v(\tilde{l}) = \tilde{v}_0. \tag{76}$$

In order to simplify the calculation, we perform a simple transformation $l' = l - \tilde{l}$ which makes the flow start at $l' = 0$. The new scaling equations describing the classical regime will be

$$\frac{\mathrm{d}[T(l')\Gamma^{-1}(l')]}{\mathrm{d}l'} = 2[T(l')\Gamma^{-1}(l')] \tag{77}$$

$$\frac{\mathrm{d}\mu(l')}{\mathrm{d}l'} = 2\mu(l') - \frac{1}{2\pi}\frac{\Lambda^2 v(l')}{\frac{\hbar^2\Lambda^2}{2m} - \mu(l')} \tag{78}$$

$$\frac{\mathrm{d}v(l')}{\mathrm{d}l'} \simeq 2v(l') - \frac{5m^2}{\pi\hbar^4}\frac{v^2(l')}{\Lambda^2}. \tag{79}$$

Equation (79) has been obtained neglecting $\mu(l')$ in equation (74). In this form the equation can be solved and we obtain the exact solution

$$v(l') = \frac{2\tilde{v}_0}{B\tilde{v}_0 + (2 - B\tilde{v}_0)\exp(-2l')} \tag{80}$$

where $B = 5m^2/2\pi\hbar^4\Lambda^2$. We define l'_*, the value of l' for which we stop the scaling, by a similar condition as in the low temperature case,

$$v(l'_*) = 1 \tag{81}$$

which gives

$$\exp(2l'_*) = \frac{2 - B\tilde{v}_0}{B\tilde{v}_0 - 2\tilde{v}_0} \tag{82}$$

and from this equation we calculate

$$l'_* \simeq \frac{1}{2}\ln\left(\frac{1}{\tilde{v}_0}\right) \tag{83}$$

where we used $u\,T(l'_*) = 1$.

Equation (78) for the chemical potential can also be solved using for $v(l')$ expression (80) and we get

$$\mu(l') = \mathrm{e}^{2l'}\left[\tilde{\mu}_0 - \frac{2m}{\pi h^2 B}l' - \frac{m}{\pi\hbar^2 B}\ln\left(\mathrm{e}^{-2l'} + \frac{B\tilde{v}_0}{2 - B\tilde{v}_0}\right)\right] \tag{84}$$

where $\tilde{\mu}_0$ is given by

$$\tilde{\mu}_0 = \frac{m}{\pi\hbar^2 B}l'_* + \frac{m}{\pi\hbar^2 B}\ln\left(\mathrm{e}^{-2l'_*} + \frac{B\tilde{v}_0}{2 - B\tilde{v}_0}\right). \tag{85}$$

Using this equation $\mu(l')$ is approximated as

$$\mu(l') \simeq \mathrm{e}^{2l'}\left[\frac{2m\tilde{v}_0}{\pi\hbar^2}(l'_* - l')\right]. \tag{86}$$

5.2.2. Density of bosons. In order to calculate the bosonic density, we will use the general equations

$$n = -\frac{\partial F}{\partial \mu} \tag{87}$$

where the free energy will be written as

$$F(l') = \frac{1}{2\pi}\int_0^{l'} \mathrm{d}x\ \mathrm{e}^{-2x} T(x) \ln\left\{1 - \exp\left[-\frac{\Gamma(x)}{k_\mathrm{B}T(x)}\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(x)\right)\right]\right\}. \tag{88}$$

Using the relations

$$n = -\int_0^{l'_*} \mathrm{d}l'\ \frac{\partial}{\partial\mu(l')}\left(\frac{\partial F}{\partial l'}\right)\frac{\mathrm{d}\mu(l')}{\mathrm{d}\mu} \tag{89}$$

and for $\mu(l')$ the simple approximation

$$\mu(l') = \mu(\tilde{l})\mathrm{e}^{2(l'+\tilde{l})} \tag{90}$$

where $\mu(\tilde{l}) \simeq \mu$, equation (89) becomes

$$n = -\int_0^{l'_*} \mathrm{d}l'\ \mathrm{e}^{2(l'+\tilde{l})}\ \frac{\partial}{\partial\mu(l')}\left(\frac{\partial F(l')}{\partial l'}\right). \tag{91}$$

From equations (88) and (91) we calculate the general equation for bosonic density n as

$$\frac{n}{T} = \frac{1}{2\pi}\int_0^{l'_*} \mathrm{d}l_1\ \frac{\Gamma(l_1)/k_\mathrm{B}T(l_1)}{\exp\left\{\Gamma(l_1)/k_\mathrm{B}T(l_1)\left[\frac{\hbar^2\Lambda^2}{2m} - \mu(l_1)\right]\right\} - 1}. \tag{92}$$

This integral will be approximated in the high temperature domain as

$$\frac{n}{T} \simeq \frac{1}{2\pi}\int_0^{l'_*} \mathrm{d}l_1\ \frac{2m}{\hbar^2\Lambda^2}\left[1 + \frac{\hbar^2\Lambda^2}{2m}\mu(l_1) + \cdots\right] \simeq \frac{1}{2\pi}\frac{2m}{\hbar^2\Lambda^2}l'_* \tag{93}$$

and from equation (83) we calculate

$$\frac{n}{T} \simeq \frac{m}{\pi\hbar^2}\ln\left(\frac{1}{\tilde{u}_0}\right) \tag{94}$$

which is the density of bosons in the classical regime. This result gives us the possibility to perform a matching with the low temperature regime. Indeed, using $\tilde{u}_0 \sim 1/\ln T_0$ for $T_0 \ll 1$ we re-obtain the expression for $n \sim T \ln(\ln(1/T))$, the well-known result from [7].

6. Discussions

The model of a d-dimensional dilute Bose gas is extensively used in the analysis of physical properties of the BEC observed experimentally in the vapour of alkali metals, adsorbed polarized hydrogen on ^{4}He surface and the physics of the normal state in the cuprate superconductors. Our paper is complementary to that of Kolomeyski and Straley [5] which studied the system at $T = 0$. The RG method for a $d = 2$ dilute Bose gas at finite temperature has been applied by Fisher and Hohenberg [8] taking into consideration the quantum and classical regions. However, in [8] a nonregular renormalized coupling constant has been used to calculate different physical quantities. We showed that the generalization of the method given in [5] at finite temperature gives correct results for $d < 2$ and $d = 2$, in the limit of low and high temperatures.

The study of the free energy [35, 36] gives us the possibility to calculate the coherence length as a function of temperature and the specific heat in the critical region. For the interesting case $d = 2$ we calculated a critical temperature T_{2D} which is the critical temperature of QC transition. The agreement with the t-matrix calculation [7] obtained also in [8] demonstrated that the Popov [8] method of calculation is equivalent to the RG method. This result has been recently re-obtained by the Strinati group [37–39] performing a careful analysis of the diagrams from the t-matrix expansion of the $d = 2$ Bose gas. Here we confined our investigations to the $d < 2$ model which presents universal properties [40] and the $d = 2$ model which is non-universal. Indeed, the number of condensed bosons, the coherence length and the specific heat depend on the energy scale $T_0(\Lambda)$. These results represent a starting point in the application of the RG method for a trapped Bose gas. On the other hand, the existent experimental data [30–33] showed the existence of the QC transition predicted in [7] and [5], but more accurate measurements have to be performed to confirm the critical behaviour of the coherence length and specific heat. These kinds of measurements on the cuprate superconductors are also promising near the quantum critical point.

The case of $d = 3$ Bose gas has been studied in [20–23] using the RG method obtaining for the critical exponent $\nu = 0.67$, which is in agreement with the measured values in ^{4}He experiments. A similar result can be obtained also from our calculations, but as in [20–23], only in the high temperature limit, when the new effective coupling constant is $v = k_B Tu$ and $z = 0$.

In the low temperature limit for $d = 3$ and $z = 2$ only the Gaussian fixed point is stable. We expect that the RG method, considered for confined interacting Bose gas, may give a more realistic description.

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